of OAO', or OAO' produced so as to form a complete line, is less than a semi-line. By so much the more is the segment which joins C to any point of OBO' or OAO' less than a semi-line. Consider the class of points D such that the proper segment CD does not cut either OAO' or OBO'. Consider also the class of points E such that the segment CE cuts one of the semi-lines or passes through the vertices O and O'. Every point of the surface, except such as are situated on the given semi-lines, occurs once and but once in one of these sets. Moreover from Theorem 16 it is evident that any two of the points D may be connected by a segment which does not cut either of the semi-lines and that any two of the points E may also be so connected (although in this case the segment cannot always be taken less than a semi-line) and that no point D can be connected with a point E by a segment which does not cut one of these given semi-lines. Hence the theorem is proved.

It has been seen that the points in one of the portions of the surface have the property that any segment drawn through them and intercepted by the given semi-lines are all less than a semi-line. This portion of the surface may naturally be said to be the interior of the figure and may be said to be enclosed by the broken line $OAO'BO$. We should note at this point that we have actually defined the words "interior" and "enclosed by." Our definitions are based upon properties of our figures proved from the axioms laid down. This is not usually done. It is customary to assume all such properties. We may proceed along the same lines to the following theorems.

Theorem 18. *A proper triangle divides the surface of the sphere into two parts, an interior and an exterior, such that a segment drawn through any point of the interior and intercepted by the side of the triangle is less than a semi-line, and any two points on the exterior may be connected by a segment which does not cut any side of the triangle.*

Theorem 19. *If segments are drawn to join two vertices of a triangle to points of the opposite sides they meet within the triangle.*

The proof of the former of these theorems is left to the reader, that of the latter depends on Theorem 16. Let ABC be the triangle and A', B' two points of the sides opposite A, B, respectively. Consider the triangle ACA'. The line BB' cuts the side AC by hypothesis and cannot cut the side CA' because BC is less than a semi-line. Hence it cuts AA' and the theorem is proved. There is considerable interest attaching to this theorem owing to the fact that it is often taken to be the definition of a triangle.*

A set of segments AB, BC, CD, DE, ..., is said to form a broken line. The broken line is *simple* when it does not pass twice through the same point. It is *closed* when the last point of the last segment coincides with the first point of the first segment. It if *convex* when no line on the surface cuts it in more than two points; otherwise it is *re-entrant*. A closed simple broken line may be called a *polygon*.

*See Schur, Ueber die Grundlagen der Geometrie, *Mathematische Annalen*, Vol. 55, p. 268, postulate 7; and Peano, Sui fondamenti della Geometria, *Rivista di Matematica*, Vol. 4, p. 55, et seq.

Theorem 20. *It is impossible to draw a triangle two of whose sides are semi-lines and if two of the sides are greater than semi-lines the triangle is not simple.*

Theorem 21. *If one side of a simple triangle is greater than a semi-line the triangle is improper and re-entrant.*

In the latter theorem let ABC be the triangle and AC the side which is not proper. Form the triangle which has for sides AB and BC and the proper segment drawn between A and C. This is a proper triangle and by Theorem 16 any line which cuts the sides AB and BC cannot cut the third side, the proper segment AC. Hence the line must cut the improper segment AC in two points and the given triangle is reëntrant. This is the reason that in working with triangles on the sphere the triangle is understood to be proper unless the contrary is specifically stated.

Theorem 22. *If two points of a broken line are on opposite sides of a line l, the broken line cuts the line l in at least one point and in at least two points if the broken line is closed.*

Let $ABC.....EFG$ be the broken line and P, Q two of its points which are situated on opposite sides of the line l. These points lie on certain of the sides of the broken line, say BC and EF. Consider the points P, C, D, ..., E, Q. Separate these points into three classes such that those in the first lie on the same side as P, those in the second on the same side as Q, and those in the third on the line itself. In case there is a point belonging to the third class the theorem is proved for the broken line cuts the line l at this point. In case there is no such point examine the sequence of points P, C, D, ..., E, Q. Since the first and last of this sequence are situated on opposite sides of the line l there must be two successive points of the sequence which lie on opposite sides of l, and the segment which joins these points must cut l. Hence the broken line cuts l. If the broken line be closed, as $ABC.....FGA$, it may be divided into two broken lines $PCD.....EQ$ and $QFGABP$ to each of which the foregoing reasoning applies. Hence the broken line cuts the line l in at least two points.

Theorem 23. *A convex polygon lies entirely on one side of or along the line formed by producing a side of the polygon so as to form a complete line; entirely within or along the contour of the figure formed by producing two of the sides until they intersect; and entirely within or along the contour of the triangle formed by producing three of the sides until they intersect.*

Theorem 24. *A convex polygon divides the surface into two portions, an interior and an exterior, such that any two points of the interior may be joined by a proper segment which does not cut the polygon, any two points of the exterior may be joined by a segment not necessarily proper which does not cut the polygon, and no point of the interior can be joined to any point of the exterior by a segment which does not cut the polygon.*

These theorems are demonstrated by methods so like those already given that the details may be omitted. In case of reëntrant polygons the theorems become somewhat more complicated and are of decidedly less value in elementary work. The following theorems may be stated.

Theorem 25. *Any polygon of a finite number of sides divides the surface into portions, an interior and an exterior, such that any two points of the interior or of the exterior may be joined by a broken line which does not cut the polygon, and no point of the interior can be joined to any point of the exterior by a broken line which does not cut the polygon.*

It was stated at the outset of this lecture that intimately associated with the division of the surface was the troublesome question of the arrangement of the directions issuing from a point. The solution of the difficulty depends on Theorem 17 by means of which it will be possible to prove

Theorem 26. *The directions issuing from a point may be arranged in a natural order, with respect to all lines which do not pass through the point, by associating each direction with the point in which it cuts a given line not passing through the point and by assigning to the directions the order which the points associated with them have.**

Let O be the point from which the directions issue and O' the antipodal point. The directions from O and O' each cut in one and only one point any arbitrary line l not passing through O and conversely to each point of the line corresponds one and only one direction obtainable by joining the point to O. Set up a similar correspondence between the directions issuing from O and the points of a second line l'. As the points of the lines l and l' are associated in a one to one manner with the directions issuing from the point O, they must be associated in a one to one manner with each other. Therefore, if a point describes one line and passes once and only once over each point of the line, then the corresponding point of the other line will describe the line in such a manner as to pass once and only once over each point. It remains to show that the order of description is the same. We shall show that the words "lie between" have the same significance for both lines. Let $OAA'O'$ be a direction which cuts l in A and l' in A'. Let B, C, D be three points of l and B', C', D' the three corresponding points of l'. If C lies in that segment BD which does not contain A, it will be said to be between B and D; and similarly for the corresponding points. The semi-lines $OBB'O'$ and $ODD'O'$ divide the surface into two parts in one of which A lies and in the other of which C is found if it be between B and D. The semi-line $OCC'O'$ lies in the same portion of the surface as C does. Hence C and C' lie in the same part with each other. But A and C are in different parts and hence A' and C' are in different parts. Therefore C' lies between B' and D'. Hence l and l' are described in the same or in the opposite order. As the order on the two lines is independent we may say that the order ACD on one is the same as the order $A'C'D'$ on the other, and the theorem is proved.

Theorem 27. *In case a convex polygon (including the proper triangle and the figure formed by two semi-lines which connect a pair of antipodal points) is subjected to a motion, the resulting figure is a convex polygon of which the interior corresponds to the interior of the original polygon.*

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*The directions being related in a one to one manner to the points on a line and possessing the same order may be said to form a continuum just as the points on a line do.

Theorem 28. *If the point O remains fixed during a motion the directions issuing from O change their positions without suffering a change in order.*

The former theorem is left to the reader; the latter may be proved as follows. Let OAO' and OBO' be two directions between which the direction OCO' lies. Let the three directions be moved into $OA'O'$, $OB'O$, and $OC'O'$, respectively. The region enclosed by OAO' and OBO' is, by the foregoing theorem, moved into the region enclosed by $OA'O'$ and $OB'O'$. Hence if OC lies in the original region, OC' lies in the transformed region. The words "lie between" have, therefore, the same significance before and after the motion. Hence the order is not changed or it is reversed.

To rule out the second supposition consider a line l on the surface and let the directions issuing from O be associated with the points on l as described under Theorem 26. Let OA be moved into OA' and suppose that the intersections of these directions with the line l be $\overline{A}$ and $\overline{A}'$. Let OX be a variable direction issuing from O and cutting l in $\overline{X}$. Now cause $\overline{X}$ to describe continuously the line l from $\overline{A}$ to $\overline{A}'$. If OX' be the direction which after the motion corresponds to OX the point $\overline{X}'$ will describe the line l from $\overline{A}'$ towards $\overline{A}$. Hence at some position $\overline{X}$ and $\overline{X}'$ will coincide and to this point will correspond a fixed direction issuing from O.* This is impossible for then there would be no motion (Theorem 1). Hence the order of the directions issuing from a point cannot be changed by a motion. The theorem is proved. And from this point we can proceed in the next lecture to develop further the theory of angles and in particular to the establishment of the existence and properties of right angles.

*This conclusion is based on two steps which we shall not prove here but which ought to be mentioned and may be left to the reader to demonstrate. First, if one of two rigidly connected directions OA and OB is moved continuously about the point O, the other moves continuously about that point. Second, if two points X and Y, starting from two points A and B, move continuously in opposite directions until X falls on B, then they must have coincided at some point.

A METHOD OF TRANSVECTION IN THE ACTUAL COEFFICIENTS, AND AN APPLICATION TO EVECTANTS.

By O. E. GLENN, University of Pennsylvania.

The most common methods of obtaining, in actual coefficients, the concomitants of a binary algebraic form become almost impracticable in the case of forms of high order. If the concomitants are given in symbolic form, the process consists in performing tedious algebraic reductions, and combination of the symbols involved, in order to reduce them back to the coefficients which they were originally taken to represent. The other methods are processes of differentiation, the most important of which is the annihilator. The nature of the former method limits it to forms of low order, while the latter, which figures prominently in the methods of Cayley and Sylvester, involves the theories of partition of numbers and indeterminate analysis.

In this paper is briefly sketched a theory wherein every concomitant is represented as a summation. The advantages of the method are marked in the cases of the Aronhold and other invariant processes and in the theory of evectants. An interesting deduction from the theory is a proof that the complete system for a form is coextensive with transvectants of the form over other of its transvectants. This proof is not, however, included in the present note.

Consider two binary forms, of orders m and n, respectively, $m \geq n$:

$$f = a_x^m = a_0 x_1^m + m a_1 x_1^{m-1} x_2 + \cdots\cdots$$

$$\phi = b_x^n = b_0 x_1^n + n b_1 x_1^{n-1} x_2 + \cdots\cdots$$

If the symbol

$$_p\left\{ \begin{matrix} q \\ \mu \end{matrix} \right\}_{st} \left\{ \begin{matrix} u \\ \nu \end{matrix} \right\}_w \equiv {}_pI^q{}_{st}I^u{}_w$$

be used to denote the following sum of binomial products

$$\left\{ \begin{matrix} q \\ p \end{matrix} \right\} \left\{ \begin{matrix} u \\ t \end{matrix} \right\} + \left\{ \begin{matrix} q \\ p+1 \end{matrix} \right\} \left\{ \begin{matrix} u \\ t+1 \end{matrix} \right\} + \cdots + \left\{ \begin{matrix} q \\ s \end{matrix} \right\} \left\{ \begin{matrix} u \\ w \end{matrix} \right\},$$

then the rth transvectant of f over ϕ will be

$$(f, \phi)_r = {}^*\binom{r}{\lambda} [{}_0I^k{}_{00}I^l{}_0 + {}_1I^k{}_{00}I^l{}_1 + {}_2I^k{}_{00}I^l{}_2 + \cdots + {}_{n-r}I^k{}_{00}I^l{}_{n-r} + {}_{n-r+1}I^k{}_{10}I^l{}_{n-r} +$$

$$\cdots + {}_{m-r}I^k{}_{m-n,\,0}I^l{}_{n-r} + {}_{m-r}I^k{}_{m-n+1,1}I^l{}_{n-r} +$$

$$\cdots + {}_{m-r}I^k{}_{m-r,\;n-r}I^l{}_{n-r}] a_{\mu+\lambda} b_{r+\nu-\lambda} (x_1 x_2)^{m+n-2r},$$

in which $k = m-r$, $l = n-r$.

*The signs of the numbers $\binom{r}{\lambda}$ alternate plus and minus.

Then if we let $[j-c]$ stand for the positive values (zero included) of the difference $j-c$ of two positive integers, as j takes successive values, the latter expression may be written*

$$(1) \qquad (f,\ \phi)_r = e_x^{m+n-2r}$$

$$= \binom{r}{\lambda} \sum_{j=0}^{j=m+n-2r} I_{[j+r-m][j+r-n][j+r-m]} I_{j-[j+r-n]}\, a_{\mu+\lambda} b_{r+\nu-\lambda}\, x_1^{m+n-2r-j} x_2^{j}.$$

It is to be noted that the subscript j is identical with the subscript of c in the term of (1) which a given j produces. This fact leads to an important modification in the cases when ϕ is a transvectant of f, or where both f and ϕ in (1) are replaced by transvectants.

Evidently we have

$$(f,f)_r = d_x^{2(m-r)} = \binom{r}{\lambda} \sum_{j=0}^{j=2(m-r)} I_{[j+r-m][j+r-m][j+r-m]} I_{j-[j+r-m]} a_{\mu+\lambda} a_{r+\nu-\lambda}\, x_1^{2(m-r)-j} x_2^{j}.$$

Then $(f,\ d)_s = e_x^{3m-2(r+s)} = (f,\ (f,f)_r)_s$

$$= \binom{s}{\lambda'} \sum_{j'=0}^{j'=3m-2(r+s)} {}_{l_1'} I_{l_2',\, j'-l_1'}\, I_{j'-l_2'}\, a_{\mu'+\lambda'} \binom{r}{\lambda} \sum_{j=s+\nu'-\lambda'} {}_{l_1} I_{l_2\, j-l_1}\, I_{j-l_2}\, a_{\mu+\lambda} a_{r+\nu-\lambda}\, x_1^{3m-2(r+s)-j'} x_2^{j'}.$$

where $l_1' = j' - [j+s-m],\ l_1 = j - [j+r-m],\ l_2' = [j'+s+2r-2m],\ l_2 = [j+r-m]$.

Replacing j by its value, we get

$$(f,d)_s = \binom{s}{\lambda'} \sum_{j'=0}^{j'=3m-2(r+s)} {}_{l_1'} I_{l_2',\, j'-l_1'}\, I_{j'-l_2'}\, a_{\mu'+\lambda'} \binom{r}{\lambda}_{k_1-l_1'} I_{l_1'' l_1'} I_{k_1-l_1''}$$

$$\times a_{\mu+\lambda} a_{r+\nu-\lambda}\, x_1^{3m-2(r+s)-j'} x_2^{j'}.$$

Finally, making use of the commutative properties, we write this

$$\binom{s}{\lambda'}\binom{r}{\lambda} \sum_{j'=0}^{j'=3m-2(r+s)} {}_{l_1'} I_{l_2',\, j'-l_1'}\, I_{j'-l_2';\, k_1-l_1'} I_{l_1'',\, l_1'}\, I_{k_1-l_1''}$$

$$\times a_{\mu'+\lambda'}\, a_{\mu+\lambda} a_{r+\nu-\lambda}\, x_1^{3m-2(r+s)-j'} x_2^{j'}$$

where $k_1 = s+\nu'-\lambda'$, $l_1' = l_1'' = [k+r-m]$.
The tth transvectant of f over this covariant will be

*The superscript to I is omitted, being fully determinate from the subscripts.

$$(f,\ l)_t=(_\lambda^t)(_\lambda^s)(_\lambda^r)\sum_{j''=0}^{j''=4m-2(t+r+s)} {}_{i_1}I_{i_2'',\,j''-i_1,\,j''-i_2''}\,I_{k_2-t_2'',\,t_2''}\,I_{k_2-t_2'',\,k_1-t_1'}\,I_{t_1''t_1'}\,I_{k_1-t_1''}$$
$$\times a_{\mu''+\lambda''}\,a_{\mu'+\lambda'}\,a_{\mu+\lambda}\,a_{r+\nu-\lambda}\,x_1^{\,4m-2(r+s+t)-j''}x_2^{\,j''}.$$

in which

$$i''_1=j''-[j''+r-m],\qquad k_2=t+\nu''-\lambda'',$$
$$i''_2=[j''+t+2r+2s-3m],\quad l'_2=[k_2+s-m],\quad l''_2=[k_2+s+2r-2m].$$

In general, we will have therefore,

$$(2)\qquad (f,\ \ldots\ldots(f,\ (f,\ (f,f)_r)_s)_t\ldots\ldots)_z=$$

$$\left\{\begin{matrix} z \\ \lambda^{(h)}\end{matrix}\right\}\left\{\begin{matrix} y \\ \lambda^{(h-1)}\end{matrix}\right\}\ldots\ldots\left\{\begin{matrix} s \\ \lambda'\end{matrix}\right\}\left\{\begin{matrix} r \\ \lambda\end{matrix}\right\}\sum_{j^{(h)}=0}^{j^{(h)}=(h+2)m-2(r+s+\ldots+z)} {}_{i_1h}I_{i_2h,\,j^{(h)}-i_1h}\,I_{j^{(h)}-i_2h;\,kh-l'h}\,I_{l'h,\,l''h}\,I_{kh-l''h}\ldots\ldots$$
$$\ldots\ldots {}_{k_1-l_1'}I_{l''_1l'_1}\,I_{k_1-l_1''}\,a_{\mu^{(h)}+\lambda^{(h)}}\,a_{\mu^{(h-1)}+\lambda^{(h-1)}}\ldots\ldots a_{\mu''+\lambda''}\,a_{\mu'+\lambda'}\,a_{\mu+\lambda}\,a_{r+\nu-\lambda}$$
$$\times x_1^{\,(h+2)m-2(r+\ldots+z)-j^{(h)}}x_2^{\,j^{(h)}}.$$

This is a covariant of order $(h+2)m-2(r+s+\ldots\ldots+z)$ degree $(h+2)$, weight $(h+2)m-(r+s+\ldots\ldots+z)$ and index $(r+s+t+\ldots\ldots+z)$.

Any concomitant in the series obtained by giving to h all integral values from zero to $\frac{2}{m}(r+s+\ldots\ldots+z-m)$, inclusive, may be obtained from the summation, directly, without knowledge of the concomitants which precede.

If $h=\frac{2}{m}(r+s+\ldots\ldots+z-m)$, (2) is an *invariant* and there being but one term, $j^{(h)}=0$, the summation sign may be removed, giving,

$$(3)\quad \left\{\begin{matrix} z \\ \lambda^{(h)}\end{matrix}\right\}\left\{\begin{matrix} y \\ \lambda^{(h-1)}\end{matrix}\right\}\ldots\ldots\left\{\begin{matrix} s \\ \lambda'\end{matrix}\right\}\left\{\begin{matrix} r \\ \lambda\end{matrix}\right\}_{kh-l'hl''h}I_{l'hl''h}I_{kh-l''h}\ldots\ldots {}_{k_1-l_1'}I_{l''_1l'_1}I_{k_1-l_1''}$$
$$\times a_{\lambda^{(h)}}\,a_{\mu^{(h-1)}+\lambda^{(h-1)}}\ldots\ldots a_{\mu+\lambda}\,a_{r+\nu-\lambda}.$$

For the sake of illustration, a few concomitants in actual coefficients are obtained.

Let $f=a_x{}^3=a_0x_1^{\,3}+3a_1x_1^{\,2}x_2+\ldots\ldots$, $\phi=b_x{}^2=b_0x_1^{\,2}+2b_1x_1x_2+\ldots\ldots$ Then

$$(f,\ \phi)=(_\lambda^1)\sum_{j=0}^{j=3} {}_{i_1}I^2{}_{i_2,\,j-i_1}I^1{}_{j-i_2}\,a_{\mu+\lambda}\,b_{1+\nu-\lambda}\,x_1^{\,3-j}x_2^{\,j}\quad (\begin{matrix} i_1=j-[j-2] \\ i_2=j-i_1\end{matrix})\quad =c_x{}^3$$

$$=(a_0b_1-a_1b_0)x_1^{\,3}+[2(a_1b_1-a_2b_0)+(a_0b_2-a_1b_1)]x_1^{\,2}x_2+[(a_2b_1-a_3b_0)$$
$$+(a_1b_2-a_2b_1)]x_1x_2^{\,2}+(a_2b_2-a_3b_1)x_2^{\,3}.$$

$$(f,\ c)_3=(_\lambda^3)a_\lambda\,(_\lambda^1)_{k_1-l_1'}\,I_{l''_1l'_1}\,I_{k_1-l_1''}\,a_{\mu+\lambda}\,b_{1+\nu-\lambda}$$
$$=a_0(a_2b_2-a_3b_1)-a_1(a_1b_2-a_2b_1)+a_2(a_0b_2-2a_2b_0+a_1b_1)-a_3(a_0b_1-a_1b_0).$$

Let next $f=a_x{}^4=a_0x_1^{\,4}+4a_1x_1^{\,3}x_2+\ldots\ldots$

$$\therefore (f,f)_4=(_\lambda^4)a_\lambda\,a_{4-\lambda}=2(a_0a_4-4a_1a_3+3a_2^{\,2})=i,$$
$$(f,\ (f,f)_2)_4=(_\lambda^4)a_\lambda\,(_\lambda^2)_{k_1-l_1'}\,I_{l''_1l'_1}\,I_{k_1-l_1''}a_{\mu+\lambda}\,a_{2+\nu-\lambda}$$

$$=2a_0(a_2a_4-a_3^2)-a_1[2(a_2a_3-2a_3a_2+a_4a_1)+2(a_1a_4-2a_2a_3+a_3a_2)]$$
$$+a_2[(a_2^2-2a_3a_1+a_4a_0)+4(a_1a_3-2a_2^2+a_3a_1)+(a_0a_4-2a_1a_3+a_2^2)]$$
$$-a_3[2(a_1a_2-2a_2a_1+a_3a_0)+2(a_0a_3-2a_1a_2+a_2a_1)]+2a_4(a_0a_2-a_1^2)$$
$$=6(a_0a_2a_4+2a_1a_2a_3-a_2^3-a_0a_3^2-a_1^2a_4)=J.$$

The non-vanishing (even) transvectants of the binary 11^{-ic} over itself are five in number. Of the forty-two transvectants of the 11^{-ic}, $f=0$, over these five, viz.,

$$F,\ G,\ \ldots\ldots,\ Z,\ A',\ B',\ \ldots\ldots,\ R',\ S',$$

K' is linear (evanescent) and N, W, F' N' and R' are of order 11. The 11th transvectants of f over these latter five give five invariants I_1, ..., I_5. These may be given in their simplified form. We have*

$$E=(f,f)_{10}=\binom{10}{\lambda}\sum_{j=0}^{j=2} {}_{j-[j-1]}I_{[j-1][j-1]}I_{j-[j-1]}\ a_{\mu+\lambda}\ a_{10+\nu-\lambda}\ x_1^{2-j}\ x_2^{j}$$

$$=(a_0a_{10}-20a_1a_9+90a_2a_8-240a_3a_7+420a_4a_6-252a_5^2)x_1^2$$
$$+2(a_0a_{11}-9a_1a_{10}+35a_2a_9-75a_3a_8+90a_4a_7-42a_5a_6)x_1x_2$$
$$+(2a_1a_{11}-20a_2a_{10}+90a_3a_9-240a_4a_8+420a_5a_7-252a_6^2)x_2^2.$$

$$I_5=(f,(f,(f,f)_{10})_1)_{11}=\binom{11}{\lambda'}a_{\lambda'}\binom{1}{\lambda}_{k_2-l'_2}I_{l'_2l'_2}I_{k_2-l'_2}\ a_{\mu'+\lambda'}\binom{10}{\lambda}_{k_1-l'_1}I_{l'_1l'_1}I_{k_1-l'_1}$$
$$\times a_{\mu+\lambda}\ a_{10+\nu-\lambda}$$

$$=-2a_0^2a_{11}^2+44a_0a_1a_{10}a_{11}-80a_0a_2a_{10}^2+360a_0a_3a_9a_{10}-960a_0a_4a_8a_{10}$$
$$+1680a_0a_5a_7a_{10}-1008a_0a_6^2a_{10}-140a_0a_2a_9a_{11}+300a_0a_3a_8a_{11}$$
$$-360a_0a_4a_7a_{11}+168a_0a_5a_6a_{11}-162a_1^2a_{10}^2+2060a_1a_2a_9a_{10}-2700a_1a_3a_8a_{10}$$
$$+3240a_1a_4a_7a_{10}-1512a_1a_5a_6a_{10}-80a_1^2a_9a_{11}+360a_1a_2a_8a_{11}-960a_1a_3a_7a_{11}$$
$$+1680a_1a_4a_6a_{11}-1008a_1a_5^2a_{11}-3600a_1a_3a_9^2+9600a_1a_4a_8a_9-1680a_1a_5a_7a_9$$
$$+10080a_1a_6^2a_9-2450a_2^2a_9^2+26700a_2a_3a_8a_9-12600a_2a_4a_7a_9+5880a_2a_5a_6a_9$$
$$-3600a_2^2a_8a_{10}+9600a_2a_3a_7a_{10}-16800a_2a_4a_6a_{10}+10080a_2a_5^2a_{10}$$
$$-43200a_2a_4a_8^2+75600a_2a_5a_7a_8-45360a_2a_6^2a_8-11250a_3^2a_8^2$$
$$+142200a_3a_4a_7a_8-12600a_3a_5a_6a_8-43200a_3^2a_7^2+75600a_3a_4a_6a_9$$
$$-45360a_3a_5^2a_9-201600a_3a_5a_7^2+120960a_3a_6^2a_7-16200a_4^2a_7^2$$
$$+367920a_4a_5a_6a_7-201600a_4^2a_6a_8+120960a_4a_5^2a_8-211680a_4a_6^3$$
$$+123480a_5^2a_6^2-211680a_5^3a_7.$$

Index=weight=22.

Now $(E_x^2, E_x^2)_2=((f,f)_{10}, (f,f)_{10})_2=D_{Ex^2}$

$$=\binom{2}{\lambda'}\binom{10}{\lambda}_{\lambda'-[\lambda'-1]}I_{[\lambda'-1][\lambda'-1]}\ I_{\lambda'-[\lambda'-1]}a_{\mu+\lambda}\ a_{10+\nu-\lambda}\binom{10}{\lambda}_{2-\lambda'-[1-\lambda']}I_{[1-\lambda'][1-\lambda']}I_{2-\lambda'-[1-\lambda']}$$
$$\times a_{\mu+\lambda}\ a_{10+\nu-\lambda}.$$

But $\binom{2}{\lambda'}\binom{10}{\lambda}=\binom{1}{\lambda'}\binom{11}{\lambda'}$. Hence there immediately results $I_5=D_{Ex^2}$.

[To be completed in the May number of the MONTHLY.]

*In a memoir on the octic form, in preparation, I give for the first time the complete system for the octic in actual coefficients.

SIMON'S CLAIM FOR GAUSS IN NON-EUCLIDEAN GEOMETRY.

By DR. GEORGE BRUCE HALSTED, Gambier, Ohio.

In Vol. X, of THE AMERICAN MATHEMATICAL MONTHLY, p. 186, foot-note, Professor J. W. A. Young quotes, with apparent acceptance, a legendary assertion of Herr Simon. This assertion is the triple blunder that there is an "established recognition" by Gauss "about 1792" that the parallel axiom is indemonstrable; and that Gauss "influenced" John Bolyai and Lobachevski, the real founders of non-Euclidean Geometry.

One of the very greatest creations in mathematics since ever the world began is, beyond peradventure, the non-Euclidean geometry.

By whom was this given to the world in print?

By a Magyar, Bolyai János, who created it in 1823, and by a Russian, Lobachevski, who had made the discovery by 1826.

Were either of these men influenced, prompted, helped, or incited by Gauss, or by any suggestion emanating from Gauss?

No; quite the contrary. Our warrant for saying this with final and overwhelming authority is the eighth volume of Gauss's own works, first published in 1900, where with great minuteness every scrap is published which could, by any interpretation, connect Gauss with the great Bolyai-Lobachevski creation.

The geometric part opens, p. 159, with Gauss's letter of 1799 to Bolyai Farkas the father of John (Bolyai János), which I gave in 1896 in my Bolyai (Vol. 3 of the Neomonic Series) as demonstrative evidence that in 1799 Gauss was still trying to prove Euclid's the only non-contradictory system of geometry, and also the system of objective space.

The first is false; the second can never be proven.

So far was Gauss even then from recognizing that the parallel axiom is not a logical necessity, that both he and his friend kept right on working away to prove it a logical necessity, and the more hot-headed of the two, Farkas, finally thought he had succeeded, and in 1804 sent his proof to Gauss, in his "Göttingen Theory of Parallels."

Gauss's judgment on this is the next thing given (pp. 160-162). He shows the weak spot, saying: "Could you *prove*, that $dkc=ckf=fkg$, etc., then were the thing perfect. However, *this theorem is indeed true*, only difficult, without already presupposing the theory of parallels, to prove rigorously."

These words knock out forever the claim made by Simon for Gauss.

Thus in 1804, instead of having or giving any light, Gauss writes that the only link missing in his friend's attempt to prove the parallel axiom a logical necessity is true, though difficult to establish without petitio principii.

Of course since Bolyai János and Lobachevski all the world now knows it is impossible to prove, knows that Gauss was mistaken. Yet both the friends continue their strivings after this impossibility.

In this very letter Gauss says: "I have indeed yet ever the hope that those

rocks sometime, and indeed before my end, will allow a thorough passage."

Farkas on December 27, 1808, writes to Gauss: "Oft thought I, gladly would I, as Jacob for Rachel serve, in order to know the parallels founded even if by another." "Now just as I thought it out on Christmas night, while the Catholics were celebrating the birth of the Saviour in the neighboring church, yesterday wrote it down, I send it to you enclosed herewith." "Tomorrow must I journey out to my land, have no time to revise; neglect I it now, may be a year is lost, or indeed find I the fault, and send it not, as has already happened with hundreds, which I as I found them took for genuine. Yet it did not come to writing those down, probably because they were too long, too difficult, too artificial; but the present I wrote off at once. As soon as you can, write me your real judgment."

This letter Gauss never answered, and never wrote again until 1832, a quarter of a century later, when the non-Euclidean geometry had been published by both Lobachevski and Bolyai János.

This settles now forever all question of Gauss having been of the slightest or remotest help or aid to young János, who in 1823 announced to his father Farkas in a letter still extant, which I saw in Maros-Vásárhely, his creation of the non-Euclidean geometry as something undreamed of in the world before.

This immortal letter, a charming and glorious outpouring of pure young genius, I gave in the Introduction to my *Bolyai*, 1896. It was reproduced in fac simile as frontispiece to the Bolyai Memorial Volume in 1902.

The equally complete freedom of Lobachevski from the slighest idea that Gauss had ever meditated anything different from the rest of the world on the matter of the parallel axiom I showed in *Science*, Vol. IX, No. 232, pp. 813-817.

Of two utterly worthless theories of parallels Gauss gave extended notices in the Göttingische gelehrte Anzeigen.

To Lobachevski's Theory of Parallels, to John Bolyai's marvelous Science Absolute of Space, Gauss vouchsafed never one printed word.

As Staeckel gently remarks, this certainly contributed thereto, that the worth of this mathematical gem was first recognized when János had long since finished his earthly career.

A PROOF THAT FOUR LINES IN SPACE ARE IN GENERAL MET BY TWO OTHER LINES.

By DR. T. M. PUTNAM, University of California.

Using the ordinary method of descriptive geometry a point in space is represented by *two* points in a plane, viz., by its horizontal and vertical projections, the vertical plane being thought of as rotated into coincidence with the horizon-

tal plane. Similarly a line in space will be represented in general by *two* lines in the plane. If two lines intersect in space their projections meet in points lying on a line perpendicular to the ground line. Let

$$y=m_ix+c_i \text{ and } y=h_ix+k_i \quad (i=1, 2, 3, 4)$$

be the projections of four arbitrary lines in space, the ground line being taken as the X-axis. Cut them respectively by the lines $y=Mx+C$ and $y=Hx+K$. If the abscissas of the corresponding points of intersection are equal, the space line whose projections are $y=Mx+C$ and $y=Hx+K$ will cut the four given lines. These conditions are

$$\frac{c_i-C}{M-m_i}=\frac{k_i-K}{H-h_i} \quad (i=1, 2, 3, 4)$$

or $Mk-HC-m_iK+h_iC+c_iH-k_iM-h_ic_i+m_ik=0$.

These four equations have in general *two* independent sets of solutions for for M, K, C, H, showing, therefore, that two lines meet the four space lines.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

99. Proposed by C. H. JUDSON,* Professor of Mathematics, Furman University, Greenville, S. C.

Seven persons met at a summer resort and agreed to remain as many days as there are ways of sitting at a round table, so that no one shall sit twice between the same two companions. They remained fifteen days. It is required to show in what way they may have been seated.

Solution by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

ABCDEFG,	AGDBCEF,	ABDEGFC,
ADFEBGC,	ADBFCEG,	ADEBCGF,
AEGBDCF,	AEDGFBC,	ADCGEFB,
ACEDFBG,	ABECGEF,	AEFCBGD,
AFBEGDC,	AEBDFCG,	ABGFDCE.

I believe that all solutions for seven persons may be obtained from that above by permutations of letters.

*March, 1899; March 1900. The general problem is treated by group theory by L. E. Dickson in an article offered February, 1904, to the *Annals of Mathematics*. A third solution for six persons is there given.

For six persons there appear to be two independent solutions, the one previously given by Dr. Judson, and the following:

ABCDEF,	ACEBDF,
ABDCFE,	ACBEFD,
ABEDFC,	ADECBF,
ABFECD,	ADBFCE,
ACDFBE,	AEDBCF.

192. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, O.

What is the difference between the squares of the two *infinite* continued fractions $\left(3+\frac{1}{6+\text{etc.}}\right)$ and $\left(2+\frac{1}{4+\text{etc.}}\right)$?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and L. E. NEWCOMB, Los Gatos, Cal.

Denote the value of the continued fractions by x and y.

Then $x-3=\frac{1}{6+x-3}=\frac{1}{x+3}$, $\therefore x^2-9=1$, $x^2=10$, $x=\sqrt{10}$;

$$y-2=\frac{1}{4+y-2}=\frac{1}{y+2},\ \therefore y^2-4=1,\ y^2=5.\ y=\sqrt{5}.$$

$\therefore x^2-y^2=5=$required result.*

194. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

In the determination of the canonical forms of Abelian transformations modulo p, one is led to the type $[b_1, b_2, b_3]$:

$$\xi_1'=\xi_1,\ \eta_1'=b_1\xi_1+\eta_1+b_2\xi_2+\eta_2+b_3\xi_3+\eta_3,\ \xi_2'=\xi_2-\xi_1,$$
$$\eta_2'=\eta_2+b_2\xi_2,\ \xi_3'=\xi_3-\xi_1,\ \eta_3'=\eta_3+b_3\xi_3.$$

Find its period and determine the conditions under which it is conjugate with $[c_1, c_2, c_3]$ under Abelian transformation.

Solution by PROPOSER.

By mathematical induction, we verify that the kth power of $[b_1, b_2, b_3]$ is

$$\xi_1'=\xi_1,\ \eta_1'=[kb_1-\tfrac{1}{6}k(k^2-1)(b_2+b_3)]\xi_1$$
$$+\eta_1+\tfrac{1}{2}k(k+1)(b_2\xi_2+b_3\xi_3)+k\eta_2+k\eta_3,$$
$$\xi_2'=\xi_2-k\xi_1,\ \eta_2'=\eta_2+kb_2\xi_2-\tfrac{1}{2}k(k-1)b_2\xi_1,$$
$$\xi_3'=\xi_3-k\xi_1,\ \eta_3'=\eta_3+kb_3\xi_3-\tfrac{1}{2}k(k-1)b_3\xi_1.$$

*Solutions based on the following interpretations are desirable. ED.

$$3+\cfrac{1}{6+\cfrac{1}{12+\cfrac{1}{24+\text{etc.}}}} \qquad 2+\cfrac{1}{4+\cfrac{1}{8+\cfrac{1}{12+\text{etc.}}}} \qquad 3+\cfrac{1}{6+\cfrac{1}{9+\cfrac{1}{12+\text{etc.}}}} \qquad 2+\cfrac{1}{4+\cfrac{1}{6+\cfrac{1}{12+\text{etc.}}}}$$

Hence if $p>3$, $[b_1, b_2, b_3]$ is of period p. For $p=2$ or 3, the period is p^2.

The question of conjugacy will be reduced to a discussion of simultaneous congruences. We assume that b_2 and b_3 are not both zero. We seek the linear substitutions S,

$$\xi_i' = \sum_{j=1}^{3} (a_{ij}\xi_j + \gamma_{ij}\eta_j), \quad \eta_i' = \sum_{j=1}^{3} (\beta_{ij}\xi_j + \delta_{ij}\xi_j) \quad (i=1, 2, 3),$$

for which $[b_1, b_2, b_3]S = S[c_1, c_2, c_3]$. The conditions are

(1) $\gamma_{11}=\gamma_{12}=\gamma_{13}=\gamma_{21}=\gamma_{31}=0$, $a_{12}+a_{13}=\delta_{21}+\delta_{31}=0$, $\delta_{2i}+\delta_{3i}=\delta_{11}$, $c_2\gamma_{2i}=\delta_{21}$,

(2) $b_i\gamma_{2i}=b_i\gamma_{3i}=-a_{1i}$, $b_i(\delta_{21}+\delta_{2i})=c_2a_{2i}$, $b_i(\delta_{31}+\delta_{3i})=c_3a_{3i}$, $c_3\gamma_{3i}=\delta_{31}$,

(3) $b_1\delta_{i1}-\beta_{i2}-\beta_{i3}=c_ia_{i1}$, $c_1a_{1i}+c_2a_{2i}+c_3a_{3i}+\beta_{2i}+\beta_{3i}=b_i(\delta_{11}+\delta_{1i})$,

(4) $a_{22}+a_{23}=a_{32}+a_{33}=a_{11}$, $c_1a_{11}+c_2a_{21}+c_3a_{31}+\beta_{12}+\beta_{13}+\beta_{21}+\beta_{31}=b_1\delta_{11}$,

where $i=2, 3$. Now c_2 and c_3 cannot both be zero, since this would require that all coefficients of η_1 in S should vanish, whereas its determinant $\neq 0$. Transforming by $(\xi_2\xi_3)(\eta_2\eta_3)$, if necessary, we may take $c_3\neq 0$, $b_3\neq 0$. For brevity we may exclude the special case $b_2=0$. Hence, by (1) and (2),

$$\delta_{21}=-c_2b_2^{-1}a_{12}=c_2b_3^{-1}a_{12}=c_3b_2^{-1}a_{12}=-c_3b_3^{-1}a_{12}.$$

If $a_{12}\neq 0$, then $b_2=-b_3$, $c_2=-c_3$, a case here excluded. Hence $a_{12}=0$, $\delta_{21}=0$. Then $a_{13}=\delta_{31}=0$, and every $\gamma_{ij}=0$. By (2), $\delta_{2i}=b_i^{-1}c_2a_{2i}$, $\delta_{3i}=b_i^{-1}c_3a_{3i}$. *The substitutions S transforming $[b_1, b_2, b_3]$ into $[c_1, c_2, c_3]$ have therefore the matrix*

(5)
$$\left(\begin{matrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ a_{21} & 0 & a_{22} & 0 & a_{23} & 0 \\ \beta_{21} & 0 & \beta_{22} & b_2^{-1}c_2a_{22} & \beta_{23} & b_3^{-1}c_2a_{23} \\ a_{31} & 0 & a_{32} & 0 & a_{33} & 0 \\ \beta_{31} & 0 & \beta_{32} & b_2^{-1}c_3a_{32} & \beta_{33} & b_3^{-1}c_3a_{33} \end{matrix}\right),$$

subject only to conditions (3), (4), and $\delta_{2i}+\delta_{3i}=\delta_{11}$, the latter becoming now

(6) $$c_2a_{2i}+c_3a_{3i}=b_i\delta_{11}.$$

We next require that matrix (5) shall be Abelian (Jordan, *Traité*, p. 172; Dickson, *Linear Groups*, p. 89). The Abelian conditions all reduce to the following:

(7) $a_{11}\delta_{11}=1$, $b_2^{-1}c_2a_{22}\beta_{32}-b_2^{-1}c_3a_{32}\beta_{22}+b_3^{-1}c_2a_{23}\beta_{33}-b_3^{-1}c_3a_{33}\beta_{23}=0$,

(8) $c_ib_2^{-1}a_{i2}^2+c_ib_3^{-1}a_{i3}^2=1$, $b_2^{-1}a_{22}a_{32}+b_3^{-1}a_{23}a_{33}=0$,

(9) $\delta_{11}a_{i1}+\delta_{12}a_{i2}+\delta_{13}a_{i3}=0,\ \delta_{11}\beta_{i1}+\delta_{12}\beta_{i2}+\delta_{13}\beta_{i3}$
$-b_2^{-1}c_ia_{i2}\beta_{12}-b_3^{-1}c_ia_{i3}\beta_{13}=0,$

where $i=2, 3$. Adding the two equations (6) and applying $(4)_1$, we get $(c_2+c_3)a_{11}=(b_2+b_3)\delta_{11}$. Then by $(7)_1$, $\delta_{11}^2=(c_2+c_3)/(b_2+b_3)$. This fraction must therefore be a quadratic residue modulo p. Suppose this condition satisfied, so that δ_{11} and a_{11} are determined except in sign. Then (6) and $(4)_1$ give

(10) $a_{23}=a_{11}-a_{22},\ c_3a_{32}=b_2\delta_{11}-c_2a_{22},\ c_3a_{33}=b_3\delta_{11}-c_2a_{11}+c_2a_{22}.$

Each of the conditions (8) then reduces to either of the forms

(11) $(b_2^{-1}+b_3^{-1})a_{22}^2-2b_3^{-1}a_{11}a_{22}+b_3^{-1}a_{11}^2=c_2^{-1},\ [(b_2+b_3)a_{22}-b_2a_{11}]^2$

$$=\frac{b_2+b_3}{c_2+c_3}\cdot\frac{b_2b_3c_3}{c_2}.$$

We have the further necessary condition that $b_2b_3c_2c_3$ shall be a quadratic residue modulo p. With this condition satisfied, a_{22} is determined by (11), and a_{23}, a_{32}, a_{33} by (10). There remains ten conditions (3), $(4)_2$, $(7)_2$, and (9) on the thirteen quantities δ_{12}, δ_{13}, a_{21}, a_{31}, β_{ij} $(i, j=1, 2, 3)$. By (6), we may write $(3)_2$ thus

(12) $$\beta_{2i}+\beta_{3i}=b_i\delta_{1i}.$$

Applying (6), (9) and (12), condition $(4)_2$ becomes

(13) $b_2\delta_{12}^2+b_2\delta_{11}\delta_{12}+b_3\delta_{13}^2+b_3\delta_{11}\delta_{13}-2\delta_{11}(\beta_{12}+\beta_{13})-c_1+b_1\delta_{11}^2=0.$

Since $\delta_{11}\neq0$, we may employ (9) to determine a_{21}, a_{31}, β_{21}, β_{31} in terms of the remaining quantities. Conditions $(3)_1$ then become

(14) $$\beta_{i2}+\beta_{i3}=a_{11}c_i(\delta_{12}a_{i2}+\delta_{13}a_{i3})\quad(i=2, 3).$$

One of these may be dropped since their sum is, by (6), the same as the sum of the pair (12). Then (6), and (14) for $i=2$, give

(15) $\beta_{32}=b_2\delta_{12}-\beta_{22},\ \beta_{23}=a_{11}c_2(\delta_{12}a_{22}+\delta_{13}a_{23})-\beta_{22},$
$\beta_{33}=b_3\delta_{13}+\beta_{22}-a_{11}c_2(\delta_{12}a_{22}+\delta_{13}a_{23}).$

Substituting these in $(7)_2$, we find that the coefficient of β_{22} is zero by (6), and that the remaining terms cancel. Hence the system of ten conditions is equivalent to the system (13), (15), (9). The latter may be solved in the order named. In particular, if $p\neq2$, we may take δ_{1i}, a_{i1}, β_{2i}, β_{3i} $(i=2, 3)$ all zero and determine $\beta_{12}+\beta_{13}$ by (13), and β_{i1} by (9). Hence $[b_1, b_2, b_3]$ and $[c_1, c_2, c_3]$ are conjugate under Abelian transformation if and only if $(c_2+c_3)\div(b_2+b_3)$ and $b_2b_3c_2c_3$ are both quadratic residues modulo p.

GEOMETRY.

215. Proposed by M. J. NEWELL, A. M., Evanston High School, Evanston, Ill.

Construct geometrically a right triangle, given the bisectors of the acute angles.

Solution by L. E. DICKSON, Ph. D., The University of Chicago.

In the absence of a purely geometrical construction, I show that the triangle is uniquely determined, a leg a being given by a quartic equation having a single positive root. Denote the acute angles of the right triangle by 2α and 2β, their bisectors by Q and P, the opposite sides by a and b, the hypotenuse by h. Then $\alpha+\beta=45°$. By a familiar theorem on bisectors,

$$P^2=2a^2h\div(a+h) \quad \ldots\ldots(1).$$
$$\therefore h=Pa^2\div(2a^2-P^2) \quad \ldots\ldots(2).$$

Now $b=Q\cos\alpha=Q\cos(45°-\beta)=Q(\cos\beta+\sin\beta)\div\sqrt{2}$. Hence

$$b\sqrt{2}=Q\left(\frac{a}{P}+\frac{ab}{P(a+h)}\right), \text{ or } b\left(\frac{P}{Q}\sqrt{2}-\frac{a}{a+h}\right)=a \quad \ldots\ldots(3).$$

Substituting in (3) the value of h from (2), we get

$$b\left(\frac{P}{Q}\sqrt{2}-\frac{2a^2-P^2}{2a^2}\right)=a \quad \ldots\ldots(4).$$

Similarly,

$$a\left(\frac{Q}{P}\sqrt{2}-\frac{2b^2-Q^2}{2b^2}\right)=b \quad \ldots\ldots(5).$$

Eliminating b between (4) and (5), we get

$$\left[\frac{P}{Q}\sqrt{2}-1+\frac{P^2}{2a^2}\right]\left[\frac{Q}{P}\sqrt{2}-1+\frac{Q^2}{2a^2}\left(\frac{P}{Q}\sqrt{2}-1+\frac{P^2}{2a^2}\right)^2\right]=1.$$

Multiplying by $16a^8PQ$ and expanding, we obtain

$$-16\sqrt{2}\,(P^2+Q^2-\sqrt{2}\,PQ)a^8+(16\sqrt{2}\,P^4-48P^3Q+32\sqrt{2}\,P^2Q^2-16Q^3P)a^6$$
$$+3(2\sqrt{2}\,P-2Q)^2P^3Qa^4+3a^2(2\sqrt{2}\,P-2Q)P^5Q^2+P^7Q^3=0 \quad \ldots\ldots(6).$$

[As a check we note that for $a=b$, $P^2=Q^2=2(2-\sqrt{2})a^2$ and (6) is satisfied]. The coefficient of a^6 equals $16P(\sqrt{2}\,P-Q)(P^2+Q^2-\sqrt{2}\,PQ)$. Set $\kappa=16(P^2+Q^2-\sqrt{2}\,PQ)$ and $\lambda=(\sqrt{2}\,P-Q)P$. Then (6) becomes

$$a^8-\tfrac{1}{2}\lambda\sqrt{2}\,a^6-6\sqrt{2}\,\kappa^{-1}\lambda^2PQa^4-3\sqrt{2}\kappa^{-1}\lambda P^4Q^2a^2-\tfrac{1}{2}\sqrt{2}\,\kappa^{-1}P^7Q^3=0 \quad \ldots\ldots(7).$$

Let $P\gtrless Q$, so that λ and κ are positive. There being a single change of sign in (7), there is one and but one positive root by Descartes' Rule of Signs.

A more elementary but less fortunate method consists in using (1) and the corresponding relation $Q^2=2hb^2\div(b+h)$ .....(8).

Now from (1), $h(2a^2-P^2)=P^2a$. But $h^2=a^2+b^2$. Hence

$$b^2=4a^4(P^2-a^2)\div(2a^2-P^2)^2 \text{ .....(9).}$$

Eliminating h between (2) and (8), we get

$$P^4a^2(2b^2-Q^2)^2=Q^4b^2(2a^2-P^2)^2.$$

In this we substitute the value of b^2 from (9) and obtain an equation of the sixth degree for a^2. Set $a=2a^2-P^2$. Then

$$(P^4+Q^4)a^6+2P^4(P^2+Q^2)a^5+P^6(2Q^2-P^2)a^4-2P^8(2P^2+Q^2)a^3$$
$$-P^{10}(2Q^2+P^2)a^2+2P^{14}a^2+P^{16}=0 \text{ .....(10).}$$

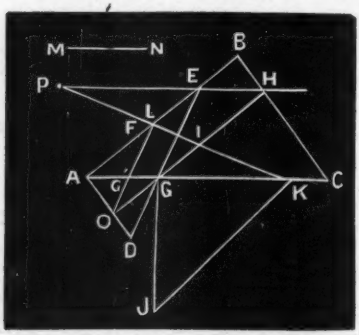
We may take $P\gtrless Q$. Then by Descartes' Rule of Signs, there are two or no positive roots. There are two positive roots, so that (10) does not uniquely determine the leg a.

218. Proposed by O. W. ANTHONY, DeWitt Clinton High School, New York City.

From a given triangle cut off an area equivalent to a given square by a line passing through a given point without the triangle.

IV. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

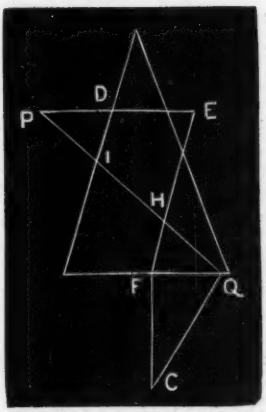
Let ABC be the given triangle, MN a side of the given square, P the given point. Through P draw PE parallel to AC meeting AB in E. Perpendicular to AB draw $AD=MN$, lay off $AF=MN$. Join DE and draw FO parallel to DE, cutting AC in G'. Draw OGH parallel to AB. At G erect $GJ=PE$ perpendicular to AC and draw $JK=PH$. Draw $PLIK$.



From similar triangles AED and AFO, we have $AE:AD=AF(=AD):AO$. $\therefore AD^2=AE\times AO=$area $AEHG$; $JK^2-JG^2=GK^2$ or $PHJ-PEL=GIK=LEHI$. $\therefore ALM=MN^2$.

V. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let ABC be the triangle and P the given point without. Draw PE parallel to AB, cutting AC in D. Make parallelogram $DEFA=$given square. On F erect the perpendicular $FG=PD$, and make $GQ=PE$. Connect P with Q, then will PQ be the required line.



For $\frac{\triangle FHQ}{\triangle PHE}=\frac{FQ^2}{PE^2}=\frac{PE^2-PD^2}{PE^2}=1-\frac{PD^2}{PE^2}$
$=1-\frac{\triangle PDI}{\triangle PEH}$;

$\therefore \triangle FHQ=\triangle PEH-\triangle PDI=DIHE$; $\therefore \triangle FHQ+IHFA=DIHE+IHFA=DEFA$, or $AIQ=DEFA$.

Also solved by L. E. Newcomb, Los Gatos, California.

219. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

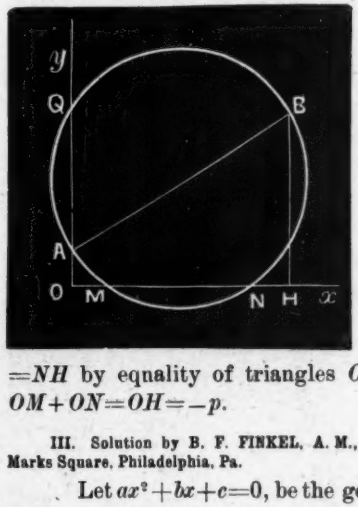
Devise a simple geometric solution of the general quadratic equation.

I. Remark by W. W. LANDIS.

A solution may be found in Klein's *Vorträge über ausgewahlte Fragen der Elementargeometrie*, pp. 28-31; in Beman and Smith's translation, p. 34.

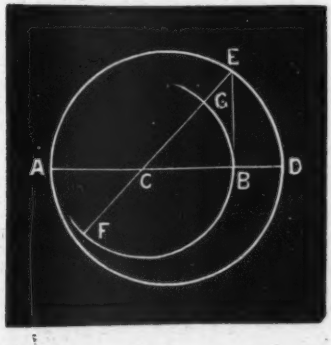
II. Solution reported by the PROPOSER.

The elegant solution by Lill (reported without proof by d'Ocagne at the Second International Congress of Mathematicians, Paris, 1900) is so simple that the Proposer has used it in his courses in elementary algebra. For the graphic solution of $x^2+px+q=0$, choose two perpendicular lines Ox and Oy, lay off unit length OA on Oy, length OH on Ox containing $-p$ units (to right or left of O, according as $-p$ is $+$ or $-$), length HB on parallel to Oy containing q units. If the circle on AB as diameter cuts Ox at M and N, then OM and ON, on the same scale, are the required roots. In proof, let Q be the second point of intersection of the circle Oy, then $OQ=HB$, since $OHBQ$ is a rectangle; $OM=NH$ by equality of triangles OQM and HBN. Hence $OM.ON=OA.OQ=q$, $OM+ON=OH=-p$.



III. Solution by B. F. FINKEL, A. M., M. Sc., 204 St. Marks Square, Philadelphia, Pa.

Let $ax^2+bx+c=0$, be the general quadratic. On the line AD, lay off $AB=2$ units, and $BD=c/2a$. On AD as a diameter describe the circle AED. At B erect the perpendicular BE. With E as a center and a radius equal to $b/2a$, describe an arc intersecting AD, or AD produced, in C. Then with C as a center and a radius equal to CB describe the circle FBG intersecting EC in G and F in the order



E, G, C, F. Then EG is equal to the value of one root of the quadratic and EF is equal to the other. For

$$BC=\sqrt{CE^2-BE^2}=\sqrt{CE^2-AB.BD}=\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}}=\frac{1}{2a}\sqrt{b^2-4ac}=CG=CF.$$

Then $EG=\dfrac{-b+\sqrt{(b^2-4ac)}}{2a}$ and $EF=\dfrac{-b-\sqrt{(b^2-4ac)}}{2a}$.

This construction only gives the absolute values of the real roots. The corresponding algebraic values must be assigned.

If $b^2/4a^2=c/a$, then $CE=BE$, which equals x.

If $b^2/4a^2<c/a$, the construction is impossible.

IV. Solution by A. H. HOLMES, Brunswick, Maine.

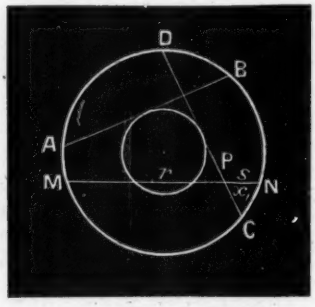
Describe a circle of radius a, and from a point on the circumference A draw tangent $AB\sqrt{b}$. Then from B draw through the center of the circle O the line BD cutting the circumference in C and D. By the principles of plane geometry the lines CB and BD will represent the unknown quantity in the quadratic equation $x^2+2ax=b$.*

V. Solution by L. LELAND LOCKE, Brooklyn, N. Y.

We have four cases to consider. These may be reduced to two.

$$\text{I.}\quad \begin{cases} x^2-px+q=0 \ldots\ldots (1), \\ x^2+px+q=0 \ldots\ldots (2); \end{cases}$$

$$\text{II.}\quad \begin{cases} x^2+px-q=0 \ldots\ldots (3), \\ x^2-px-q=0 \ldots\ldots (4). \end{cases}$$



Let $q=r.s$, r and s being any suitable factors of q.

Case I. Equations 1, 2. Draw a circumference 1, and in it a chord $AB=p$, using any convenient unit of measure. Tangent to this chord and concentric with circle 1 draw circle 2. In circle 1 place a chord MNP such that $MP=r$ and $NP=s$. Through P draw chord CD tangent to circle 2. PC and PD are the roots of the quadratics. Their values may be found, using the same unit of measure as before.

Proof. $AB=CD=p$, $CP=x_1$, $PD=x_2=p-x_1$; $CP.PD=NP.PM=s.r$,

$x_1(p-x_1)=rs$, $x_1^2-px_1+rs=0$

Similarly, $x_2^2-px_2+rs=0$ } equation 1.

*There are four cases, since a and b may each be positive or negative. The above solution suggests at once the following: Construct a circle of radius a; at the distance $\sqrt{b}$ from the diameter AB draw a parallel chord which intersects the circle (if at all) in $A'B'$. Draw $B'H$ perpendicular to AB; it is now evident that AH, HB are roots of the quadratic $(2a-x)x=b$; *i. e.* of $x^2-2ax+b=0$.

In order to solve $x^2-2ax=b$ we observe that by changing the signs of both roots the equation $x^2+2ax=b$ is obtained, and Mr. Holmes' solution applies.

Similarly, in order to solve $x^2+2ax+b=0$ we observe that its roots are the negatives of the roots of $x^2-2ax+b=0$, and the construction at the beginning of this foot-note applies. ED.

Equation 2, $x_2^2 - px_2 + q = 0$, may be solved in a similar manner by changing the sign of px and proceeding as above. The roots being the same in numerical value but opposite in sign.

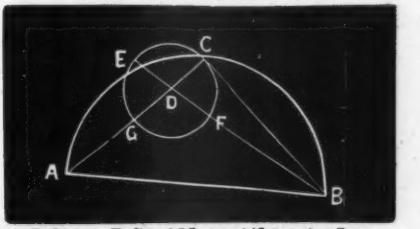
Case II. Equations 3, 4. The figure for Case II differs from that of Case I only in that P is outside of the circle 1; the points CPD being now in the order PCD. $PC = x_1$, $PD = x_2 = x_1 + p$, $PN.PM = PC.PD$; $r.s = x_1(x_1 + p)$, $x_1^2 + px_1 - rs = 0$. Similarly, $x_2^2 + px_2 - rs = 0$.

Equation 4 is solved by changing sign of px and proceeding as with equation 3, since the roots are the opposite of the roots of equation (3).

In Case I if the roots are imaginary, the point P falls within circumference 2, and the graphic method fails.

VI. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Upon AB describe a semi-circle. Let C be the mid-point of the semi-circular arc. Draw AC, BC. Let G be a point on the line AC; on GC describe a circle center D. Through D draw $BFDE$. Then taking BE, BF positive; EB, FB negative, these lines are the roots of a quadratic having $2CD$ for the coefficient of the first power of the unknown quantity, and $\sqrt{(2AB)}$ for the absolute term, the coefficient of the second power of the unknown quantity being taken unity.



Let $AB = \sqrt{2c}$, $GC = 2DC = b$.

$BE = BD + DE = BD + DC = \sqrt{(BC^2 + DC^2)} + DC = \frac{1}{2}[b + \sqrt{(b^2 + 4c)}]$.

$BF = BD - DF = BD - DC = -\frac{1}{2}[b - \sqrt{(b^2 + 4c)}]$. Taking $x^2 \pm bx = c$. then for $x^2 + bx = c$, $EB = -\frac{1}{2}[b + \sqrt{(b^2 + 4c)}]$, $BF = -\frac{1}{2}[b - \sqrt{(b^2 + 4c)}]$.

For $x^2 - bx = c$, $BE = \frac{1}{2}[b + \sqrt{(b^2 + 4c)}]$, $FB = \frac{1}{2}[b - \sqrt{(b^2 + 4c)}]$.

If c be negative, the results still hold.

Also solved by G. W. Greenwood, B. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill., by use of circle and hyperbola.

219A. Proposed by H. F. MacNEISH, A.B., Assistant in Mathematics, University High School, Chicago, Ill.

Draw a line through a given point which shall divide a given quadrilateral into two equivalent parts; (1) when the point lies in a side of the quadrilateral, (2) when the point is without, (3) within the quadrilateral.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $ABCD$ be the given quadrilateral. Produce DA, CB till they meet in F. Join AC and draw BS parallel to AC, join CS; then triangle SCD = quadri-$ABCD$. Disect SD in H, HF in G, and join CH, CG.

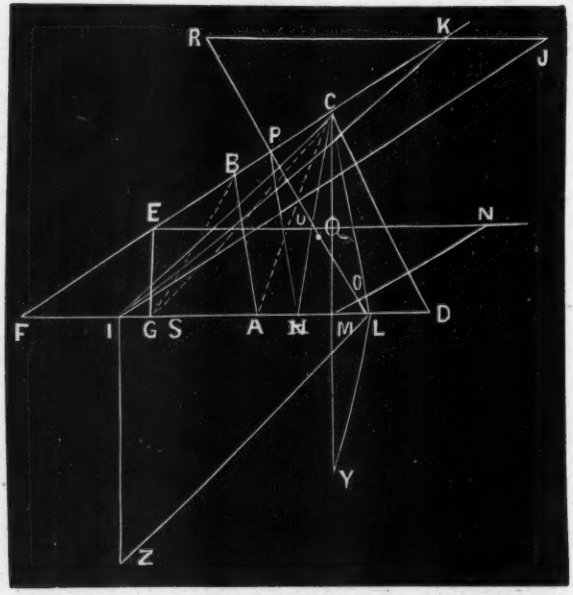
(1). Let P be the point in the side BC. Join PH and draw CL parallel to PH, join PL. Triangle PCL = triangle HCL.

$\therefore PCL + LCD = PCDL = HCL + LCD = HCD = \frac{1}{2}ABCD$.

(2). Let R be the point without the quadrilateral. Draw RK parallel to

FD meeting FC in K. Join KG and draw CI parallel to KG, join KI. Then $ICG=ICK$ and $FCI=FCI$.

$\therefore FCG=FKI$. Draw IJ parallel to FC, then parallelogram $FKJI=FCH$. At the point I draw IZ perpendicular to FD and equal to RK, draw $ZL=RJ$.



Then $RPUL$ is the line required; for $LZ^2-IZ^2=IL^2$ or $VRJ-PRK=IVL=PVJK$.

$\therefore FPL=FKJI=FCH$. Now $SBA=SBC$. $\therefore FBA=FCS$ or $FPL-FBA=FCH-FCS$.

$\therefore ABPL=SCH=\frac{1}{2}ABCD$.

(3). Let Q be the point within the quadrilateral. Draw QE parallel to FD meeting FC in E. Join EG and draw CM parallel to EG, join EM. Then $EGM=EGC$ and $FEM=FCG$. Draw MN parallel to FC; then $FENM=FCH$. At the point M draw MY perpendicular to FD and equal to EQ, draw $YL=QN$. Then $PQOL$ is the required line, for $LY^2=MY^2+ML^2$ or $QON=PEQ+MOL$.

$\therefore FPL=FENM=FCH$. $\therefore FPL-FBA=FCH-FCS=SCH$.

$\therefore ABPL=SCH=\frac{1}{2}ABCD$.

Also solved by G. W. Greenwood, B. A. (Oxon).

220. Proposed by G. B.M. ZERR, A. M., Ph. D., Parsons, West Va.

Two triangles are circumscribed to a given triangle ABC, having their sides perpendicular to the sides of the given triangle. Prove that the two triangles are equal, and find the area of these triangles.

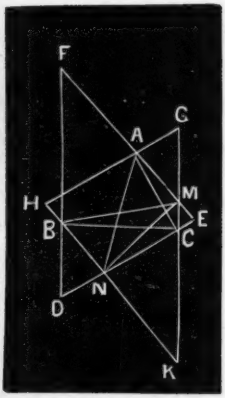
I. Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

The perpendiculars to a side of the given triangle at its extremities, which are corresponding sides of the circumscribed triangles, are symmetrical with respect to any point on the perpendicular bisector of that side. Hence the two triangles are symmetric with respect to the circumcenter of the given triangle and are therefore equal. The area of either is equal to

$$\tfrac{1}{2}(c^2\cot B+a^2\cot C+b^2\cot A)+S,$$

where S is the area of the given triangle.

II. Solution by PROPOSER.



Let ABC be the given triangle area $\triangle$; DEF, GHK the circumscribed triangles; FE, GK intersecting in M; DC, HK intersecting in N. Join BM, AN, MN. The triangles DEF, GHK are equiangular. $\angle BAM=\angle BCM=\angle ACN=\angle ABN=$ a right angle.

$\therefore$ A circle passes through A, M, C, B and A, C, N, B, respectively.

$\therefore \angle AMB=\angle ACB=\angle ANB$.

$\therefore AM=BN$ and is parallel to it, $MN=AB$ and is parallel to it.

$\therefore AF=NK$, $ME=HB$, $\therefore FE=HK$ and $DEF=GHK$. Let $A=$area of DEF, then $2A=EC.AC+FA.AB+DB.BC+\triangle=b^2\cot A+c^2\cot B+a^2\cot C+\triangle=c^2\cot A+a^2\cot B+b^2\cot C+\triangle$.

$\therefore 4A=[a^4+b^4+c^4-bc(b^2+c^2)-ac(c^2+a^2)-ab(a^2+b^2)+4\triangle^2]/2\triangle$.

Also solved by F. D. Posey, San Mateo, California; L. E. Newcomb, Los Gatos, California; and J. Scheffer, Hagerstown, Md.

MECHANICS.

165. Proposed by O. W. ANTHONY, DeWitt Clinton High School, New York City.

Find the approximate form of a tower of circular cross section 1000 feet high and having a radius of lower base 20 feet, and so constructed that all the parts of the structure shall be subject to the same stress, due to the weight of the part of the tower above.

III. Solution by CHRISTIAN HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, O.

Let w=the weight per unit of volume; s=the stress (pressure per square inch); y=the radius of cross section at distance x from top.

Then πy^2=the area of cross section at distance x from top, and, by the conditions of the problem, the increment of the area for any increment of x multiplied by s, must equal the increment of the weight, we get, in the limit, $\frac{dy}{dx}=\frac{w}{2s}y$. Integrating, we find $y=ce^{(w/2s)x}$. Now since $y=20$ when $y=1000$, $c=20e^{-(1000w/2s)}$.

$\therefore y=20e^{(w/2s)(x-1000)}$, which, by assigning proper values to w and s, will give the form of an element of the lateral surface of the tower. If we call $w=\frac{1}{10}$ and $s=20$, the radii of the tower at different elevations would be as follows: At 100 feet, 12.13 feet; at 200, 7.36; at 300, 4.46; at 400, 2.70; at 500, 1.64; at 600, .996; at 700, .604; ot 900, .222; at 990, .1417; and at top, .1348 feet or 1.6 inch.

This result will fulfill the required condition as to stress everywhere except near the top.

166. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

If a gravitating particle of mass m be placed at the point (a, b, c) prove

that the work required to move a particle of unit mass from the point (x, y, z) to an infinite distance is $m[(x-a)^2+(y-\beta)^2+(z-c)^2]^{-\frac{1}{2}}$. Prove also that

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\{m[(x-a)^2+(y-b)^2+(z-c)^2]^{-\frac{1}{2}}\}=0,$$

except when $x=a$, $y=b$, $z=c$.

Solution by S. A. COREY, Hiteman, Iowa.

Let $r=[(x-a)^2+(y-b)^2+(z-c)^2]^{-\frac{1}{2}}$=distance from point (a, b, c) to point (x, y, z). The gravitating force exerted between the two particles will then be m/r^2. As the gravitating force is always directed along r, the work required to move the unit mass from the point (x, y, z) to an adjacent point $(x+dx, y+dy, z+dz)$ by any path would be $(m/^2)dr$. The total work done as r is increased from r to infinity would be $m\int_r^\infty \frac{dr}{r^2}$, or m/r, as requirod.

$$\text{As } m\frac{d^2}{dx^2}\cdot\frac{1}{r}=m\frac{3(x-a)^2-r^2}{r^5},\quad m\frac{d^2}{dy^2}\cdot\frac{1}{r}=m\frac{3(y-b)^2-r^2}{r^5},$$

$$m\frac{d^2}{dz^2}\cdot\frac{1}{r}=m\frac{3(z-c)^2-r^2}{r^5},\quad m\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\cdot\frac{1}{r}=0,$$

for all values of r except $r=0$, *i. e.*, except when $x=y$. $y=b$, $z=c$.

Also solved by G. W. Greenwood, B.A. (Oxon); M. E. Graber, A.B.; W. W. Landis; and Proposer.

GROUP THEORY.*

1. [Miscellaneous No. 144, in January, 1904. p. 21.] Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Burnside, in his systematic search for all simple groups of orders from 661 to 1092 (*Proc. Lond. Math. Soc.*, 1895, pp. 333-338) overlooked the orders $2^3.3^2.11$ and $1008=2^4.3^2.7$. The former is immediately excluded. Discuss the latter.

Solution by W. BURNSIDE, The Croft, Bromley Road, Catford, England.

With regard to the second [order 1008], it is clear that the supposition that the group is simple, with 8 subgroups of order 7, leads to the contradiction of a group of degree 8 with operations of order 21. In the only other case, viz., 36 subgroups of order 7, each self-conjugate in a subgroup of order 28, the group must be expressible as a *primitive* group of degree 36, in which the subgroup of order 28, which leaves one symbol unchanged, has a self-conjugate operation of order 2, which leads to an obvious contradiction.

*See page 100.

MISCELLANEOUS.

144. [Transferred to Group Theory, Problem 1, p. 98.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

198. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Solve $2^{x+y}=6^y$; $3^x=3.2^{y+1}$.

199. Proposed by SAUL EPSTEEN, Ph. D., Chicago, Ill.

Solve $(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_6)=(x+a_1)(x+a_2)(x+a_3)(x+a_4)(x+a_5)(x+a_6)$.

GEOMETRY.

224. Proposed by WILLIAM HOOVER, Ph. D., Professor of Mathematics, Ohio State University, Athens, O.

The equations to two circles are $(x-a)^2+(y-b)^2=c^2$, $(x-b)^2+(y-a)^2=c^2$; give the length of their common tangent and *thence* the condition that the two circles may touch.

225. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Determine the sides of a triangle, given the lengths of (1) the three altitudes, (2) the three medians, (3) the radii of the escribed circles, (4) the radius R of the circumscribed circle and any two of the three quantities r=radius of inscribed circle, s=semi-perimeter, $\triangle$=area.

CALCULUS.

178. Proposed by SAUL EPSTEEN, Ph. D., The University of Chicago.

Evaluate $\int_0^{\frac{1}{2}\pi} \frac{d\phi}{1+\sin^2\phi}$.

MECHANICS.

168. Proposed by M. E. GRABER, A. B., Instructor in Mathematics and Physics in Heidelberg University, Tiffin, Ohio.

In Bifilar Suspension W is the weight of the suspended mass, a and b the distances between the threads above and below, h the vertical height of the threads. If the difference in vertical components of tension is n times W and θ is the angle turned through in azimuth, momental resistance is $\frac{1}{4}(1-n^2)W(ab/h)\sin\theta$. [Perry's *Engineering*.]

GROUP THEORY.

In view of the interest that is being taken in America in the Theories of Finite and Continuous Groups as well as in their various applications it is thought desirable to have a special department in THE AMERICAN MATHEMATICAL MONTHLY devoted to problems in this field. The receipt of letters from a number of our contributors suggesting this innovation leads us to hope it will be a successful one. THE EDITORS.

1. Proposed on p. 21 as Miscellaneous and solved on p. 98, as Group Theory No. 1.

2. Proposed by W. BURNSIDE, The Croft, Bromley Road, Catford, England.

Show that the group of the biquadratic equation $x^4+2ax^2+b=0$, in which a and b are rational numbers, while a^2-b is not the square of a rational number, is in general a dihedron group of order 8; but that (*i*) if b is the square of a rational number the group is a non-cyclical Abelian group of order 4; and (*ii*) if $b=a^2\div(1+n^2)$, where n is a rational number, the group is a cyclical group of order 4.

3. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Show that the equation $x^4-ax^3+bx^2-ax+1=0$, in which a and b are rational numbers, while neither $2-b+(\frac{1}{2}a)^2$ nor $(1+\frac{1}{2}b)^2-a^2$ is the square of a rational number, is irreducible in the domain R of rational numbers; determine its group for this domain.

NOTES.

Editor Dickson has been granted leave of absence for six months from the University of Chicago to accept the appointment of Research Assistant to the Carnegie Institution. Dr. Dickson's investigation will relate to the application of Group Theory to certain problems in Geometry and Function Theory. F.

School Mathematics announces the following list of associate editors: Robert J. Aley, University of Indiana; Joseph V. Collins, State Normal School at Stevens Point, Wis.; Clarence E. Comstock, Bradley Polytechnic Institute; Ellery W. Davis, University of Nebraska; David Felmley, State Normal School, Normal, Ill.; J. E. Gould, University of Washington; Franklin Turner Jones, University School, Cleveland, O.; G. A. Miller, Stanford University; W. F. Moncrief, Winthrop College; E. H. Moore, University of Chicago; Frank Morley, Johns Hopkins University; Charles W. Newhall, Shattuck School, Faribault, Minn.; H. B. Newson, University of Kansas; John C. Packard, High School, Brookline, Mass.; Arthur Schultze, New York High School of Commerce; Monroe B. Snyder, Central High School, Philadelphia; David Eugene Smith, Teachers College, Columbia University.

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SPHERICAL GEOMETRY.

By EDWIN BIDWELL WILSON.

LECTURE V. RIGHT ANGLES AND ALLIED TOPICS.

Theorem 29. *The two portions of the surface into which the surface is divided by a line are superposable, that is, congruent: and may be called hemispheres.*

Theorem 30. *All hemispheres are congruent.*

Theorem 31. *The order of three directions a, b, c, issuing from a point O is the same as the order of the three corresponding opposite directions a', b', c', issuing from that point.*

To show that two portions of surface coincide it is necessary to demonstrate that every point in one of them lies in the other, and conversely. Or, since this has presumably been done once in Theorem 27, it will merely be necessary to show that the boundaries of the portions of surface coincide, provided those boundaries are convex polygons or the other figures specified in the statement of Theorem 27. To demonstrate Theorem 29 let l be the given line, O any point of it, a and a' the two opposite directions which issue from O. Cause a and a' to rotate about the point O with the whole surface until a and a' fall respectively on the opposite directions a' and a. The line l has been moved into itself in such a way that what was originally on one side of the line has been moved to the other side. Theorem 27 will now apply. Theorem 30 is proved in a similar manner. The last of the above three theorems may also be stated by saying that if the direction b issues into the angle formed by a and c, the opposite direction issues into the vertical angle. Were the theorem untrue, the whole line formed of b and b' would lie in one hemisphere. This is impossible.